ENGINEERING MATHEMATICS I

UNIT-5

HIGHER ORDER DE

Vibha Masti

11.11.2019
Monday

HIGHER ORDER LINEAR DIFFERENTIAL EQNS

Introduction

An LDE is one in which the dependent variable and its derivatives occur only in first degree and are not multiplied together.

The general form of the linear DE of n^{th} order is

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_n y = X \longrightarrow (1)$$

where $P_1, P_2, P_3 \dots P_n$ and X are functions of x.

An LDE of n^{th} order with constant coefficients is of the form

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + k_n y = X \longrightarrow (2)$$

note: if the RHS $= 0$, the function is homogeneous

where $k_1, k_2, k_3 \dots k_n$ are constants and X is a function of x.

Consider

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + k_n y = 0 \longrightarrow (3)$$ (n^{th} order LDE - homogeneous)

General solution has n arbitrary constants.

If $y_1, y_2 \dots y_n$ are n independent solutions of (3), then

$c_1 y_1 + c_2 y_2 + \dots + c_n y_n = u$ is also a solution of the equation (3), called the complete solution.

Therefore,

$$\frac{d^n u}{dx^n} + k_1 \frac{d^{n-1} u}{dx^{n-1}} + k_2 \frac{d^{n-2} u}{dx^{n-2}} + \dots + k_n u = 0 \longrightarrow (4)$$

If $y = v$ is a particular solution of eq. (2), then

$$\frac{d^n v}{dx^n} + k_1 \frac{d^{n-1} v}{dx^{n-1}} + \dots + k_n v = X \quad \text{———} \quad (5)$$

Adding (4) and (5)

$$\frac{d^n}{dx^n}(u+v) + k_1 \frac{d^{n-1}}{dx^{n-1}}(u+v) + \dots + k_n(u+v) = X$$

This shows that the general solution of equation (2) is made up of two parts.

The function u satisfies equation (4) and is called the complementary function.

The second function v is called the particular solution or the particular integral.

complementary function

Hence, general solution = CF + PI — particular integral

The operator D

Writing $\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2 \dots \frac{d^n}{dx^n} = D^n$

Equation (2) becomes

$$D^n y + k_1 D^{n-1} y + \dots + k_n y = X$$

$$(D^n + k_1 D^{n-1} + \dots + k_n) y = X$$

$$\boxed{f(D)\, y = X}$$

RULES TO FIND CF

Consider an n^{th} order LDE which can be written in the form

$$f(D)y = X$$

To find the CF, we write $f(D)y = 0$, where $f(D)$ is a polynomial in D of degree n.

$$f(D) = 0$$

$$\Rightarrow D^n + K_1 D^{n-1} + \dots + K_n = 0 \quad (n^{th} \text{ degree polynomial})$$

CASE I

Let $m_1, m_2 \dots m_n$ be the roots.

$$\therefore (D-m_1)(D-m_2)\dots(D-m_n)y = 0$$

Consider $(D-m_n)y = 0$

$$\Rightarrow \frac{dy}{dx} - m_n y = 0 \quad (\text{Leibnitz eqn.})$$

$$IF = e^{\int -m_n dx} = e^{-m_n x}$$

The solution is

$$y e^{-m_n x} = C_n$$

$$\boxed{y = C_n e^{m_n x}}$$

Similarly,

$$y = c_1 e^{m_1 x}, \quad y = c_2 e^{m_2 x} \dots y = C_{n-1} e^{m_{n-1} x} \quad \text{are also solutions.}$$

Therefore, CF $u = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \dots + C_n e^{m_n x}$, provided the roots are all real and distinct.

CASE II

If there are 2 repeated roots, say $m_1 = m_2 = m$, and the remaining $n-2$ are all real and distinct, then the CF is given by

$$CF = u = (c_1 x + c_2)e^{mx} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$$

(Proof in Grewal)

CASE III

If there is a pair of complex roots; say

$$m_1 = \alpha + i\beta \text{ and } m_2 = \alpha - i\beta$$

and the remaining are real and distinct, then

$$CF = e^{\alpha x}\left[c_1 \cos\beta x + c_2 \sin\beta x\right] + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

CASE IV

If the complex roots $\alpha \pm i\beta$ repeat; say

$$m_1 = m_2 = \alpha + i\beta, \quad m_3 = m_4 = \alpha - i\beta$$

and the remaining are all real and distinct, then

$$CF = e^{\alpha x}\left[(c_1 x + c_2)\cos\beta x + (c_3 x + c_4)\sin\beta x\right] + c_5 e^{m_5 x} + \dots + c_n^{m_n x}$$

Example

Let roots $= 2, 2, 0, 0, 1 \pm i\sqrt{3}$

$$CF = (c_1 x + c_2)e^{2x} + (c_3 x + c_4) + e^{x}(c_5 \cos\sqrt{3} + c_6 \sin\sqrt{3})$$

Inverse Operator

(a) $D \sin x = \cos x$

(b) $\frac{1}{D} \sin x = ?$

Let $\frac{1}{D} \sin x = y \Rightarrow Dy = \sin x$

$$\frac{dy}{dx} = \sin x \Rightarrow dy = \sin x \, dx$$

$$\int dy = \int \sin x \, dx$$

$$y = -\cos x$$

Therefore:

- $D f(x) = f'(x)$
- $\frac{1}{D} f(x) = \int f(x) \, dx$

USING THE $\frac{1}{D}$ OPERATOR

$$f(D) y = X$$

The function $\frac{X}{f(D)}$ is a function of x independent of arbitrary constant.

Also, $\frac{f(D) X}{f(D)} = X$

This shows that $\frac{X}{f(D)}$ is the particular integral of the DE.

Clearly, $f(D)$ and $\frac{1}{f(D)}$ are inverse operators.

RULES FOR FINDING PARTICULAR INTEGRAL (PI)

Case I

When $X = e^{ax+b}$ a, b constants

Consider

$$(D^n + k_1 D^{n-1} + k_2 D^{n-2} + \dots + k_n)\, y = X = e^{ax+b}$$

$$f(D)\, y = e^{ax+b}$$

Now,

$$D e^{ax+b} = a e^{ax+b}$$

$$D^2 e^{ax+b} = a^2 e^{ax+b}$$

$$\vdots \qquad\qquad \vdots$$

$$D^n e^{ax+b} = a^n e^{ax+b}$$

$$\therefore (D^n + k_1 D^{n-1} + \dots k_n)\, e^{ax+b} = a^n e^{ax+b} + k_1 a^{n-1} e^{ax+b} + \dots + k_n e^{ax+b}$$

$$= e^{ax+b}\left(a^n + k_1 a^{n-1} + \dots + k_n\right)$$

$$f(D)\, e^{ax+b} = e^{ax+b} f(a)$$

$$\frac{e^{ax+b}}{f(D)} = \frac{e^{ax+b}}{f(a)}$$

$$\boxed{\frac{X}{f(D)} = \frac{X}{f(a)} = y}$$

$$\therefore PI = \frac{e^{ax+b}}{f(a)}, \text{ provided } f(a) \neq 0$$

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If $f(a)=0$,

$$PI = \frac{x e^{ax+b}}{f'(a)}, \quad f'(a) \neq 0$$

Check proof in Grewal

Case II

When $X = \sin(ax+b)$ or $\cos(ax+b)$

$$PI = \frac{X}{f(D)}$$

Replace D^2 by $-a^2$

Example:

$$\frac{\sin(2x+3)}{D^3+2D^2-D+1}$$

$a=2$
$-a^2=-4$

$$= \frac{\sin(2x+3)}{-4D-8-D+1} = \frac{\sin(2x+3)}{-5D-7}$$

Multiply by $(5D-7)$

$$\frac{(5D-7)\sin(2x+3)}{(5D-7)(-5D-7)} = \frac{(5D-7)\sin(2x+3)}{49-25D^2}$$

$$= \frac{(5D-7)\sin(2x+3)}{49+100} = \frac{5D\sin(2x+3)-7\sin(2x+3)}{149}$$

If $f(D)=0$, multiply numerator by x & differentiate denominator wrt x.

The introduced functions should not be integrated/differentiated

PROBLEMS

1. Solve $(D^2 + 2D + 1)\, y = 2e^{3x} + 5^x - \ln 2$

Consider the auxiliary equation

$$D^2 + 2D + 1 = 0$$
$$(D+1)^2 = 0$$
$$D = -1, -1$$

write all repetitions

(Case II, page 4)

$$CF = (c_1 x + c_2) e^{-x}$$

unique function for which $f(D) = 0$

$$PI = \frac{2e^{3x} + 5^x - \ln 2}{D^2 + 2D + 1}$$

] — X

] — f(D)

$5^x = e^{(\ln 5^x)} = e^{x \ln 5}$

$$= \frac{2e^{3x}}{D^2+2D+1} + \frac{e^{x \ln 5}}{D^2+2D+1} + \frac{e^0(-\ln 2)}{D^2+2D+1}$$

Here, $\dfrac{X}{f(D)} = \dfrac{X}{f(a)}$

$$= \frac{2e^{3x}}{9+6+1} + \frac{e^{x \ln 5}}{(\ln 5)^2 + (2\ln 5) + 1} - \frac{\ln 2}{1}$$

$$= \frac{e^{3x}}{8} + \frac{e^{x \ln 5}}{(\ln 5)^2 + 2\ln 5 + 1} - \ln 2$$

The complete solution is

$$y = CF + PI$$

$$y = (c_1 x + c_2)e^{-x} + \frac{e^{3x}}{8} + \frac{5^x}{(\ln 5)^2 + 2\ln 5 + 1} - \ln 2$$

learn conversion to exponential

$$\sinh x = \frac{e^x - e^{-x}}{2} \; ; \; \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

2. Solve $(D^6 - 64)y = e^x \cosh 2x$

The auxiliary equation

$$D^6 - 64 = 0$$

$$(D^3 + 2^3)(D^3 - 2^3) = 0$$

$$D^3 + 2^3 = 0$$

$$(D+2)(D^2 - 2D + 4) = 0$$

$D = -2$ $\qquad D = \dfrac{2 \pm \sqrt{4-16}}{2}$

$$= 1 \pm \sqrt{1-4} = 1 \pm \sqrt{-3}$$

$$D = 1 + i\sqrt{3}, \; 1 - i\sqrt{3}$$

$$D^3 - 2^3 = 0$$

$$(D-2)(D^2 + 2D + 4) = 0$$

$D = 2$ $\qquad D = \dfrac{-2 \pm \sqrt{4-16}}{2}$

$$= -1 \pm i\sqrt{3}$$

factors: $-2, 2, \; 1 \pm i\sqrt{3}, \; -1 \pm i\sqrt{3}$

$$CF = c_1 e^{-2x} + c_2 e^{2x} + e^x\left(c_3 \cos\sqrt{3}x + c_4 \sin\sqrt{3}x\right) + e^{-x}\left(c_5 \cos\sqrt{3}x + c_6 \sin\sqrt{3}x\right)$$

$$PI = \frac{e^x \cosh 2x}{D^6 - 64} = \frac{1}{2}\left(\frac{e^x(e^{2x} + e^{-2x})}{D^6 - 64}\right)$$

$$= \frac{1}{2}\left(\frac{e^{3x} + e^{-x}}{D^6 - 64}\right) = \frac{1}{2}\left(\frac{e^{3x}}{3^6 - 64} + \frac{e^{-x}}{(-1)^6 - 64}\right)$$

$$= \frac{1}{2}\left(\frac{e^{3x}}{665} + \frac{e^{-x}}{-63}\right) = \frac{e^{3x}}{1330} - \frac{e^{x}}{126}$$

$$y = CF + PI$$

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3. Solve $(D^3 - D^2 + 4D - 4)y = e^x$

auxiliary equation:

synthetic division

1	1	-1	4	-4
		+1	+0	+4
	1	0	4	$\boxed{0}$
	x^2	x	x^0	

$$D^3 - D^2 + 4D - 4 = 0$$

$$D^2(D-1) + 4(D-1) = 0$$

$$(D^2+4)(D-1) = 0$$

$D^2 + 4 = 0$ $\boxed{D = 1}$

$$D = \frac{\pm\sqrt{-16}}{2} = \frac{\pm 4i}{2}$$

$\boxed{D = \pm 2i}$

$$CF = (c_1 \cos 2x + c_2 \sin 2x)e^0 + c_3 e^x$$

$$PI = \frac{e^x}{D^3 - D^2 + 4D - 4}, \; D = 1 \quad (e^{ax+b}, a = 1)$$

$$= \frac{e^x}{1 - 1 + 4 - 4} = \frac{e^x}{0} \longrightarrow \text{not allowed}$$

$$PI = \frac{xe^x}{3D^2 - 2D + 4} = \frac{xe^x}{3 - 2 + 4} = \frac{xe^x}{5}$$

complete solution is

$$y = CF + PI$$

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4. $(D^2-4D+1)y = \sin^2 x$

auxiliary eq:

$$D^2-4D+1=0$$

$$D = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{4-1} = 2 \pm \sqrt{3}$$

$a \pm \sqrt{b}$ surds

$$CF = C_1 e^{(2+\sqrt{3})x} + C_2 e^{(2-\sqrt{3})x}$$

Case II page 7

$$PI = \frac{\sin^2 x}{D^2-4D+1} = \frac{1-\cos 2x}{2(D^2-4D+1)} = \frac{e^0}{2D^2-8D+2} - \frac{\cos 2x}{2D^2-8D+2}$$

$$= \frac{1}{2} - \frac{\cos 2x}{2D^2-8D+2}$$

$\sin(ax+b)$

$D^2 \to -a^2$

$a=2$

$D^2=-4$

$$= \frac{1}{2} - \frac{\cos 2x}{-8+2-8D}$$

$$PI = \frac{1}{2} - \frac{\cos 2x}{-8D-6} = \frac{1}{2} + \frac{(\cos 2x)(4D-3)}{(8D+6)(4D-3)}$$

$$= \frac{1}{2} + \frac{1}{2}\left(\frac{4D\cos 2x}{(4D)^2-9} - \frac{3\cos 2x}{16D^2-9}\right)$$

$$= \frac{1}{2} + \frac{1}{2}\left(\frac{4D\cos 2x}{-64-9} - \frac{3\cos 2x}{-64-9}\right)$$

$$= \frac{1}{2} + \frac{1}{2}\,\frac{8\sin 2x}{73} + \frac{1}{2}\,\frac{3\cos 2x}{73}$$

$$PI = \frac{1}{2}\left(1 + \frac{8\sin 2x + 3\cos 2x}{73}\right)$$

$$y = CF + PI$$

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5. $(D^2+9)\, y = \cos 2x \cos x$

$2(D^2+9)\, y = \cos 3x + \cos x$

auxiliary eqn.

$2(D^2+9) = 0$

$D = \pm 3i$

$CF = c_1 \cos 3x + c_2 \sin 3x$

$PI = \dfrac{\cos 3x + \cos x}{2D^2+18}$

$D^2 \longrightarrow -a^2$

$D^2 \longrightarrow -9$

$= \dfrac{\cos 3x}{2D^2+18} + \dfrac{\cos x}{2D^2+18}$

$= \dfrac{\cos 3x}{0} + \dfrac{\cos x}{16}$

$= \dfrac{x \cos 3x}{4D} + \dfrac{\cos x}{16}$

$\dfrac{x}{4} \int \cos 3x \, dx$

$\dfrac{x \, D \cos 3x}{4D^2} = \dfrac{-3x \sin 3x}{-12}$

$\dfrac{x}{4} \dfrac{\sin 3x}{3}$

$PI = \dfrac{x \sin 3x}{12} + \dfrac{\cos x}{16}$

$y = CF + PI$

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6. Solve $\frac{d^3y}{dt^3} + 4\frac{dy}{dt} = \sin 2t$

Let $D = \frac{d}{dt}$ (not $\frac{d}{dx}$)

$(D^3 + 4D)\,y = \sin 2t$

A.E

$D^3 + 4D = 0$

$D(D^2 + 4) = 0$

$D = 0$ or $D = \pm 2i$

$CF = (c_1 \cos 2t + c_2 \sin 2t) + c_3$

$PI = \frac{\sin 2t}{D(D^2+4)}$ $\quad a = 2 \Rightarrow -a^2 = -4$

$= \frac{\sin 2t}{0} \Rightarrow$ diff.

$= t\frac{\sin 2t}{3D^2+4} = \frac{t \sin 2t}{-8}$

$PI = -\frac{t \sin 2t}{8}$

Case III

When $X = x^m$ (plus more terms), a polynomial in x of degree m

Here, $PI = \dfrac{X}{f(D)} = \dfrac{x^m}{f(D)}$

$$= [f(D)]^{-1} x^m$$

We now expand $[f(D)]^{-1}$ in ascending powers of D as far as D^m using binomial theorem. and operate on x^m

$$(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \dots$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \dots$$

7. Solve $(D^2-1)y = 2x^4 - 3x + 1$

A.E:

$$D^2 - 1 = 0$$
$$D = \pm 1$$

$$CF = c_1 e^x + c_2 e^{-x}$$

$$PI = \frac{2x^4 - 3x + 1}{-(1-D^2)}$$

$$= -(1-D^2)^{-1}(2x^4 - 3x + 1)$$

$$= -(1 + D^2 + D^4)(2x^4 - 3x + 1)$$

$$= -(2x^4 - 3x + 1) - D^2(2x^4 - 3x + 1) - D^4(2x^4 - 3x + 1)$$

$$= -2x^4 + 3x - 1 - D(8x^3 - 3) - D^3(8x^3 - 3)$$

$$= -2x^4 + 3x - 1 - 24x^2 - D^2(24x^2 - 3)$$

$$= -2x^4 + 3x - 1 - 24x^2 - D(48x) = -2x^4 + 3x - 1 - 48 - 24x^2$$

$$PI = -2x^4 - 24x^2 + 3x - 49$$

$$y = CF + PI$$

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8. Solve $(D^2+2)y = x^3 + x^2 + e^{-2x} + \cos(3x+2)$

The auxiliary equation is

$$D^2+2=0$$
$$D = \pm\sqrt{2}\,i$$

$$CF = \left(c_1\cos(\sqrt{2}x) + c_2\sin(\sqrt{2}x)\right)$$

$$PI = \frac{x^3+x^2+e^{-2x}+\cos(3x+2)}{D^2+2}$$

$$= \underset{(PI_1)}{\frac{x^3+x^2}{D^2+2}} + \underset{(PI_2)}{\frac{e^{-2x}}{D^2+2}} + \underset{(PI_3)}{\frac{\cos 3x+2}{D^2+2}}$$

$a=3$
$-a^2=-9$

$$PI_1 = \frac{x^3+x^2}{D^2+2}$$ (case III - page 14)

$$= \frac{x^3+x^2}{2\left(1+\frac{D^2}{2}\right)} = \frac{1}{2}\left(1+\frac{D^2}{2}\right)^{-1}(x^3+x^2) = \frac{1}{2}\left(1-\frac{D^2}{2}\right)(x^3+x^2)$$

expand unit D^3

$(1+x)^{-1} = (1-x+x^2 \ldots)$

$$= \frac{1}{2}\left(x^3+x^2 - \frac{D^2}{2}(x^3+x^2)\right)$$

$$= \frac{x^3+x^2}{2} - \frac{1}{4}D\left(3x^2+2x\right)$$

$$= \frac{x^3+x^2}{2} - \frac{1}{4}(6x+2) = \frac{1}{2}\left(x^3+x^2-3x+1\right)$$

$$PI_2 = \frac{e^{-2x}}{D^2+2} = \frac{e^{-2x}}{(-2)^2+2} = \frac{e^{-2x}}{6}$$

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$$PI_3 = \frac{\cos(3x+2)}{D^2+2} = \frac{\cos(3x+2)}{-9+2} = \frac{-1}{7}\cos(3x+2)$$

$$y = CF + PI_1 + PI_2 + PI_3$$

9. Solve $(D-2)^2 y = 8(e^{2x} + \sin 2x + x^2)$

$$(D^2-4D+4)\,y = (e^{2x} + \sin 2x + x^2)8$$

$$CF = D^2-4D+4=0$$
$$(D-2)^2=0$$
$$D = 2,2$$

$$CF = c_1e^{2x}x + c_2e^{2x}$$

$$PI = \frac{(e^{2x} + \sin 2x + x^2)8}{D^2-4D+4}$$

$$= \underbrace{\frac{8e^{2x}}{D^2-4D+4}}_{PI_1} + \underbrace{\frac{(\sin 2x)8}{(D^2-4D+4)}}_{PI_2} + \underbrace{\frac{8x^2}{D^2-4D+4}}_{PI_3}$$

$$PI_1 = \frac{8e^{2x}}{0}$$

$$= \frac{(xe^{2x})(8)}{2D-4} = \frac{8e^{2x}}{0}$$

$$= \frac{(x^2e^{2x})(8)}{2} = 4x^2e^{2x}$$

Replace D^2 by -4

$$PI_2 = \frac{(\sin 2x)8}{D^2-4D+4} = \frac{8\sin 2x}{-4D} = \frac{-2\sin 2x}{D} = -2\int \sin 2x\,dx$$

$$= \frac{2\cos 2x}{2} = \cos 2x$$

$$PI_3 = \frac{8x^2}{(D-2)^2} = \frac{8x^2}{(-2)^2(1-\frac{D}{2})^2} = 2\frac{x^2}{(1-D/2)^2} = 2(1-\frac{D}{2})^{-2}x^2$$

$$(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \dots$$

$$(1-x)^{-2} = 1 + {}^{-2}C_1(x) + {}^{-2}C_2 x^2 + \dots$$

$$= 1 - (-2x) + (\frac{-2\times -3}{2!})x^2$$

$$= 1 + 2x + 3x^2$$

$$= 2(1 + D + \frac{3D^2}{4})x^2$$

$$= 2(x^2 + 2x + \frac{3}{4}D(2x)) = 2(x^2 + 2x + \frac{3}{2})$$

$$y = CF + PI_1 + PI_2 + PI_3$$

10. $$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} = 1 + x + e^{-3x}$$

Let $\frac{d}{dx} = D$

AE: $(D^3 + 3D^2) = 0$

$D^2(D+3) = 0$

$D = 0, 0, -3$

$$CF = (c_1 x + c_2) + c_3 e^{-3x}$$

$$PI = \frac{1 + x + e^{-3x}}{D^3 + 3D^2}$$

$$= \underbrace{\frac{1}{D^3+3D^2}}_{PI_1} + \underbrace{\frac{x}{D^3+3D^2}}_{PI_2} + \underbrace{\frac{e^{-3x}}{D^3+3D^2}}_{PI_3}$$

$$PI_1 = \frac{xe^{0x}}{3D^2+6D} = \frac{x^2e^0}{6D+6} = \frac{x^2}{6}$$

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go upto D^3, not D

$$PI_2 = \frac{x}{D^3+3D^2} = \frac{x}{3D^2\left(\frac{D}{3}+1\right)} = \left(1+\frac{D}{3}\right)^{-1}\frac{x}{3D^2}$$

$$(1+x)^{-1} = 1 + {}^{-1}C_1 x + {}^{-1}C_2 x^2 + \dots$$

$$= 1 - x + \quad + x^2 - x^3 + x^4 - x^5$$

$$= \left(1 - \frac{D}{3} + \frac{D^2}{9} - \frac{D^3}{27}\right)\frac{x}{3D^2}$$

$$= \left(\frac{x}{9D^2} - \frac{x}{27D} + \frac{x}{27} - \frac{Dx}{81}\right)$$

$$= \frac{x^2}{18D} - \frac{x^2}{54} + \frac{x}{27} - \frac{1}{81}$$

$$= \frac{x^3}{54} - \frac{x^2}{54} + \frac{x}{27} - \frac{1}{81}$$

$$PI_3 = \frac{e^{-3x}}{D^3+3D^2} = \frac{e^{-3x}}{-27+27} = \frac{e^{-3x}}{0}$$

$$= \frac{x\,e^{-3x}}{3D^2+6D} = \frac{x e^{-3x}}{27-18} = \frac{x e^{-3x}}{9}$$

$-1/81$ can be grouped here

omit 2 terms

$$y = \underbrace{c_1 x + c_2 + c_3 e^{-3x}}_{CF} + \underbrace{\frac{x^2}{6}}_{PI_1} + \underbrace{\frac{x^3}{18} - \frac{x^2}{18}}_{PI_2} + \underbrace{\frac{x e^{-3x}}{9}}_{PI_3}$$

$\frac{x}{27}$ can be grouped

Case IV

When $X = (e^{ax+b})V$ where $V = \sin(cx+d)$ or $\cos(cx+d)$ or $x^m + \dots$

$$PI = \frac{X}{f(D)} = \frac{e^{ax+b}V}{f(D)}$$

Shift $D \longrightarrow D+a$ (in the denominator)

$$PI = e^{ax+b}\frac{V}{f(D+a)}$$ $\longrightarrow$ then proceed with method for V

11. Solve $(D^2-2D-1)y = e^x \cos x$

AE :

$$D^2-2D-1=0$$

$$D = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$$

$D = 1+\sqrt{2}$ $\quad$ $D = 1-\sqrt{2}$

$$CF = C_1 e^{(1+\sqrt{2})x} + C_2 e^{(1-\sqrt{2})x}$$

$$PI = \frac{e^x \cos x}{D^2-2D-1} = e^x \frac{\cos x}{(D+1)^2-2(D+1)-1} = e^x \frac{\cos x}{D^2+\cancel{2D}+1-\cancel{2D}-3}$$

$$= e^x \frac{\cos x}{D^2-2}$$ replace $D^2 \longrightarrow -1$

$$= e^x \frac{\cos x}{-3} = -\frac{1}{3}e^x \cos x$$

$$y = CF + PI$$

$$= C_1 e^{(1+\sqrt{2})x} + C_2 e^{(1-\sqrt{2})x} - \frac{1}{3}e^x \cos x$$

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12. Solve $(D^2-4)y = x\sinh x$ $\qquad \sinh x = \dfrac{e^x - e^{-x}}{2}$

AE: $D^2 - 4 = 0$

$(D+2)(D-2) = 0$

$D = 2, -2$

$CF = c_1 e^{2x} + c_2 e^{-2x}$

$$PI = \frac{\frac{1}{2} x (e^x - e^{-x})}{D^2 - 4} = \frac{1}{2}\left(\frac{xe^x - xe^{-x}}{D^2-4}\right)$$

$$= \frac{1}{2}\left(\frac{e^x x}{D^2-4} - \frac{xe^{-x}}{D^2-4}\right)$$

$$= \frac{1}{2}\left(e^x \frac{x}{(D+1)^2-4} - e^{-x}\frac{x}{(D-1)^2-4}\right)$$

$$= \frac{1}{2}\Bigg(\underbrace{e^x \frac{x}{D^2+2D-3}}_{PI_1} - \underbrace{e^{-x}\frac{x}{D^2-2D-3}}_{PI_2}\Bigg)$$

$$PI_1 = \frac{1}{2}\left(e^x \frac{x}{-3\left(\frac{-D^2}{3} - \frac{2}{3}D + 1\right)}\right)$$

$$= \frac{1}{2}\left(e^x \left(1 - \left(\frac{D^2}{3} + \frac{2}{3}D\right)\right)^{-1} x \left(\frac{-1}{3}\right)\right)$$

$$= \frac{1}{2}e^x \left(1 + \frac{2}{3}D\right) x \left(-\frac{1}{3}\right)$$

$$= \frac{-e^x}{6}\left(x + \frac{2}{3}\right) = \frac{-xe^x}{6} - \frac{e^x}{9}$$

$$PI_2 = \frac{-1}{2}e^{-x}\left(\frac{x}{-3\left(\frac{-D^2}{3} + \frac{2}{3}D + 1\right)}\right)$$

$$= \frac{1}{6}e^{-x}\left(1 - \frac{2}{3}D\right)x = \frac{e^{-x}}{6}\left(x - \frac{2}{3}\right)$$

$$= \frac{xe^{-x}}{6} - \frac{e^{-x}}{9}$$

$$PI = \frac{-xe^{x}+xe^{-x}}{6} - \frac{e^{x}}{9} - \frac{e^{-x}}{9}$$

$$= \frac{-x}{3}\left(\frac{e^{x}-e^{-x}}{2}\right) - \frac{2}{9}\left(\frac{e^{x}+e^{-x}}{2}\right)$$

$$PI = \frac{-x}{3}\sinh x \quad -\frac{2}{9}\cosh x$$

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$$y = CF + PI$$

13. Solve $(D^3 - 3D^2 + 3D - 1)y = x^2 e^x$

AE: $D^3 - 3D^2 + 3D - 1 = 0$

$$(D-1)^3 = 0$$

$$D = 1, 1, 1$$

$$CF = (C_1 x^2 + C_2 x + C_3)e^x$$

$$PI = \frac{x^2 e^x}{(D-1)^3} \longleftarrow$$

product of exp
& sum
$\downarrow$
shift $D \rightarrow D+a$

$$= e^x\left(\frac{x^2}{D^3}\right)$$

$$= e^x\left(\frac{x^3}{3D^2}\right) = e^x\left(\frac{x^4}{12D}\right) = e^x\left(\frac{x^5}{60}\right)$$

$$y = CF + PI$$
$$= (C_1 x^2 + C_2 x + C_3)e^x + \left(\frac{x^5}{60}\right)e^x$$

Case V

only of first degree

When $X = xV$ where $V = \sin(ax+b)$ or $\cos(ax+b)$

Here, $PI = \dfrac{X}{f(D)} = \dfrac{xV}{f(D)}$

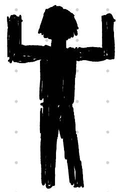
sin or cos

$$= x\frac{V}{f(D)} - \frac{f'(D)}{f(D)^2}V$$

14. $(D^2-1)y = x\sin 3x$

AE: $D^2-1=0$
$(D+1)(D-1)=0$
$D = -1, +1$

$CF = c_1e^{-x} + c_2e^{x}$



$$PI = \frac{x\sin 3x}{D^2-1}$$

$$= x\frac{\sin 3x}{D^2-1} - \frac{2D}{(D^2-1)^2}\sin 3x$$

$$= x\frac{\sin 3x}{-10} - \frac{6\cos 3x}{100}$$

$$= \frac{-x\sin 3x}{10} - \frac{6\cos 3x}{100}$$

Alternate Method for PI

$$PI = \frac{x\sin 3x}{D^2-1}$$

$e^{i\theta} = \cos\theta + i\sin\theta$

$\cos\theta = RP$ of $e^{i\theta}$
$\sin\theta = IP$ of $e^{i\theta}$

$$= \frac{IP \text{ of } x e^{3ix}}{D^2-1}$$

$$= IP \text{ of } \frac{xe^{3ix}}{D^2-1}$$

$$= IP \text{ of } e^{3ix} \frac{x}{(D+3i)^2 - 1}$$

$(1-x)^{-1} = 1 + x + x^2 + \dots$

$$= IP \text{ of } e^{3ix} \frac{x}{D^2 + 6Di - 10}$$

$$= IP \text{ of } \frac{e^{3ix}}{10} \frac{(-1)\, x}{\left(1 - \frac{6Di}{10} - \frac{D^2}{10}\right)}$$

$$= IP \text{ of } \frac{-e^{3ix}}{10} \left(1 - \left[\frac{6Di}{10} + \frac{D^2}{10}\right]\right)^{-1} x$$

$$= IP \text{ of } \frac{-e^{3ix}}{10} \left(1 + \frac{6Di}{10}\right) x$$

$$= IP \text{ of } \frac{-e^{3ix}}{10} \left(x + \frac{6i}{10}\right)$$

$$= IP \text{ of } \frac{-e^{3ix}}{10} \left(x + \frac{3i}{5}\right)$$

$$= IP \text{ of } \frac{-1}{10}(\cos 3x + i \sin 3x)\left(x + \frac{3i}{5}\right)$$

$$= \frac{-1}{10}\left(\frac{3}{5}\cos 3x + x \sin 3x\right)$$

$$PI = \frac{-3\cos 3x}{50} - \frac{x \sin 3x}{10}$$

15. Solve $(D^2 - 2D + 1)y = x \cos x$

AE: $D^2 - 2D + 1 = 0$

$(D-1)^2 = 0$

$D = 1, 1$

$CF = (c_1 x + c_2) e^x$

$$PI = \frac{x\cos x}{(D-1)^2}$$

$$= x\frac{\cos x}{(D-1)^2} - \frac{2(D-1)}{(D-1)^4}\cos x$$

$$= x\frac{\cos x}{D^2-2D+1} - \frac{2(D-1)}{(D^2-2D+1)^2}\cos x$$

replace D^2 with -1

$$= x\frac{\cos x}{-2D} - \frac{2(D-1)}{4D^2}\cos x$$

integrate

replace D^2 with -1

I don't really understand

$$= -\frac{x}{2}\sin x - \frac{(D-1)}{2(-1)}\cos x$$

$$= \frac{-x\sin x}{2} + \frac{(D-1)\cos x}{2}$$

differentiate

$$= -\frac{x\sin x}{2} + \frac{(-\sin x - \cos x)}{2}$$

If $X = e^{ax+b}\sin(cx+d)\,x^n$, use case $\underline{IV}$

16. Solve $(D^2-4D+4)y = 8x^2e^{2x}\sin 2x$

AE: $D^2-4D+4=0$

$(D-2)^2=0$

$D=2,2$

$CF = (C_1x+C_2)e^{2x}$

$D \rightarrow D+2 \Rightarrow (D-2)^2 \rightarrow D^2$

$$PI = e^{2x}\frac{8x^2\sin 2x}{D^2}$$

$$= e^{2x}(8)\frac{x^2\sin 2x}{D^2}$$

$$= 8e^{2x}\iint x^2\sin 2x\,dx$$

$u = x^2$ $\quad v = \frac{-\cos 2x}{2}$

$du = 2x\,dx$ $\quad dv = \sin 2x$

$$= 8e^{2x}\int -\frac{\cos 2x}{2}x^2 + \frac{\cancel{2}x\sin 2x}{2\times\cancel{2}} + \frac{\cancel{2}}{4}\left(+\frac{\cos 2x}{\cancel{2}}\right)$$

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$$= 8e^{2x}\left(-\frac{\sin 2x}{4}x^2 - \left(\frac{\cos 2x}{8}\right)(2x) + \left(\frac{\sin 2x}{16}\right)(2)\right)$$

$$+ 8e^{2x}\int \frac{x\sin 2x}{2}\,dx + 8e^{2x}\int \frac{\cos 2x}{4}\,dx$$

$$= -2e^{2x}\sin 2x\, x^2 - 2e^{2x}(\cos 2x)\,x + e^{2x}\sin 2x$$

$$+ 8e^{2x}\left(\left(-\frac{\cos 2x}{4}\right)x - \left(-\frac{\sin 2x}{8}\right) + \frac{\sin 2x}{8}\right)$$

$$= -2x^2e^{2x}\sin 2x - 2xe^{2x}\cos 2x + e^{2x}\sin 2x$$

$$- 2xe^{2x}\cos 2x + e^{2x}\sin 2x + e^{2x}\sin 2x$$

$$= -2x^2e^{2x}\sin 2x - 4xe^{2x}\cos 2x + 3e^{2x}\sin 2x$$

19.11.19
Tuesday

METHOD OF VARIATION OF PARAMETERS

A second order linear ODE is of the form

$$y'' + Py' + Qy = X$$

where P, Q, X are all functions of x.

In particular, if P & Q are constants, then we have

$$y'' + k_1y' + k_2y = X$$

If $D = \frac{d}{dx}$, then the auxiliary equation is

$$D^2 + k_1D + k_2 = 0$$

$$CF = c_1y_1(x) + c_2y_2(x)$$

where y_1 and y_2 are independent solutions

To find the PI by the method of variation of parameters,

$$PI = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$$

Polish Mathematician

where W is called the Wronskian of y_1 & y_2, given by

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix}$$

The complete solution (CS) = $y = CF + PI$

called variation of parameters

- PI derived from CF by varying c_1 & c_2 as functions

17. Solve $(D^2+1)y = \operatorname{cosec} x$

AE : $D^2+1 = 0$
$D = i, i$

$$CF = (c_1 \cos x + c_2 \sin x)e^0$$
$$= c_1 y_1 + c_2 y_2$$

$y_1 = \cos x$ $\qquad y_2 = \sin x$

$$\therefore W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$PI = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$$

$$PI = -\cos x \int \frac{\sin x \; X}{1}\, dx + \sin x \int \frac{\cos x \; X}{1}\, dx$$

$$= -\cos x \int dx + \sin x \int \cot x \, dx$$

$$PI = -x \cos x + \sin x \ln(\sin x)$$

$$CS: \quad y = c_1 \cos x + c_2 \sin x - x\cos x + \sin x \ln(\sin x)$$

18. $(D^2 + 2D + 2)y = e^{-x} \sec^3 x$

AE: $D^2 + 2D + 2 = 0$

$$D = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm \sqrt{1-2} = -1 \pm i$$

$$CF = (c_1 \cos x + c_2 \sin x)\, e^{-x}$$

$$y_1 = e^{-x}\cos x \qquad y_2 = e^{-x} \sin x$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x}\cos x & e^{-x}\sin x \\ -e^{-x}\sin x - e^{-x}\cos x & e^{-x}\cos x - e^{-x}\sin x \end{vmatrix}$$

$$W = e^{-2x}\cos^2 x - e^{-2x}\cancel{\sin x \cos x} + e^{-2x}\sin^2 x + e^{-2x}\cancel{\sin x \cos x}$$

$$W = e^{-2x}$$

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$$PI = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$$

$$= -e^{-x}\cos x \int \frac{\cancel{e^{-2x}} \sin x \sec^3 x}{\cancel{e^{-2x}}} dx + e^{-2x} \sin x \int \frac{\cancel{e^{-x}} \cos x \sec^3 x}{\cancel{e^{-2x}}} dx$$

$$= -e^{-x}\cos x \int \tan x \sec^2 x \, dx + e^{-2x} \sin x \int \sec^2 x \, dx$$

$$= -e^{-x}\cos x \frac{\tan^2 x}{2} + e^{-x} \sin x \tan x$$

$$= \frac{e^{-x}}{2} \left(\sin x \tan x \right)$$

19. Solve $(D^2+3D+2)y = e^{e^x}$

AE: $D^2+3D+2=0$

$$D = \frac{-3 \pm \sqrt{9-8}}{2} = \frac{-3 \pm 1}{2}$$

$-2, -1$

$$CF = c_1 e^{-2x} + c_2 e^{-x}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix}$$

$$= -e^{-3x} + 2e^{-3x} = e^{-3x}$$

$$PI = -y_1 \int \frac{y_2 X}{W}\, dx + y_2 \int \frac{y_1 X}{W}\, dx$$

$$= -e^{-2x} \int \frac{e^{-x}}{e^{-3x}} e^{e^x}\, dx + e^{-x} \int \frac{e^{-2x}\, e^{e^x}}{e^{-3x}}\, dx$$

$$= -e^{-2x} \int e^{2x} e^{e^x}\, dx + e^{-x} \int e^{x} e^{e^x}\, dx$$

$$= -e^{-2x} \int e^{x} e^{x} e^{e^x}\, dx + e^{-x} e^{e^x}$$

$$= -e^{-2x} \int t e^{t}\, dt + e^{-x} e^{e^x}$$

$$= -e^{-2x} \left[te^{t} - e^{t} \right] + e^{-x} e^{e^x}$$

$$= -e^{-2x} \left(e^{x} e^{e^x} - e^{e^x} \right) + e^{-x} e^{e^x}$$

$$= -e^{-x} \cancel{e^{e^x}} + e^{-2x} e^{e^x} + \cancel{e^{-x} e^{e^x}}$$

$$PI = e^{-2x} e^{e^x}$$

CS: $y = CF + PI$

20. $(D^2+1)\,y = \dfrac{1}{1+\sin x}$ 30

AE: $D^2+1 = 0$

$D = i, i$

CF: $(c_1 \cos x + c_2 \sin x)$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$PI = -\cos x \int \frac{\sin x}{1+\sin x}\,dx + \sin x \int \frac{\cos x}{1+\sin x}\,dx$$

$$= -\cos x \int dx + \cos x \int \frac{dx}{1+\sin x} + \sin x \ln(1+\sin x)$$

$$= -x\cos x + \sin x \ln(1+\sin x) + \cos x \int \frac{dx}{1+\sin x}$$

$$= -x\cos x + \sin x \ln(1+\sin x) + \cos x \int \frac{\sec^2 x/2\,dx}{1+2\tan x/2 + \tan^2 x}$$

$$+ \cos x \int \frac{dt}{t^2+2t+1}$$

DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

(i) Cauchy's Homogeneous Equations

An equation of the form

$$x^n \frac{d^n y}{dx^n} + k_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + k_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + k_{n-1} x \frac{dy}{dx} + k_n y = X$$

where k_i are all constants, is called Cauchy's Equation

- Homogeneous: power of x = order of derivative
- Convert to one with constant coefficients.

To reduce the above equation to a DE with constant coefficients, we write

$$x = e^t \quad \text{or} \quad t = \ln x$$

independent variable: t

Consider dy/dx term

$$\therefore \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$x\frac{dy}{dx} = \frac{dy}{dt} = Dy \qquad \text{where } D = \frac{d}{dt}$$

Consider d^2y/dx^2 term

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{x}\right)\frac{dy}{d} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$= \frac{-1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right) \cdot \frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}$$

$$\therefore\ x^2\frac{d^2y}{dx^2} = -\frac{dy}{dt} + \frac{d^2y}{dt^2}$$

$$x^2\frac{d^2y}{dx^2} = -Dy + D^2y$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

Similarly,

$$x^3\frac{d^3y}{dy^3} = D(D-1)(D-2)y$$

and so on

With this substitution, the given equation reduces to a DE with constant coefficients.

(2) LEGENDRE'S EQUATIONS

An equation of the form

$$(ax+b)^n\frac{d^ny}{dx^n} + k_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \dots + k_{n-1}(ax+b)\frac{dy}{dx} + k_n y = X$$

where k_i are all constants is called Legendre's Equation.

To reduce the above equation to a DE with constant coefficients, we write

$$ax+b = e^t \quad \text{or} \quad t = \ln(ax+b)$$

Considering dy/dx term

$$\frac{dy}{dx} = \frac{dy}{dt}\left(\frac{dt}{dx}\right) = \frac{a}{ax+b}\frac{dy}{dt}$$

$$(ax+b)\frac{dy}{dx} = a\frac{dy}{dt} = aDy$$

Considering d^2y/dt^2 term

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{a}{ax+b}\frac{dy}{dt}\right)$$

$$= \frac{dy}{dt}\left(\frac{-a^2}{(ax+b)^2}\right) + \frac{a}{ax+b}\frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$= \frac{dy}{dt}\left(\frac{-a^2}{(ax+b)^2}\right) + \left(\frac{a}{ax+b}\right)^2 \frac{d^2y}{dt^2}$$

$$(ax+b)^2\frac{d^2y}{dx^2} = -a^2\frac{dy}{dt} + a^2\frac{d^2y}{dt^2}$$

$$= a^2(-D+D^2)y$$

$$(ax+b)^2\frac{d^2y}{dx} = a^2 D(D-1)y$$

Similarly,

$$(ax+b)^3\frac{d^3y}{dx^3} = a^3 D(D-1)(D-2)y$$

and so on

22. Solve $x\dfrac{d^2y}{dx^2} - \dfrac{2}{x}y = x + \dfrac{1}{x^2}$

Multiplying by x

$$x^2\frac{d^2y}{dx^2} - 2y = x^2 + \frac{1}{x}$$

Let $t = \ln(x)$

$$x^2\frac{d^2y}{dx^2} - 2y = x^2 + \frac{1}{x}$$

$x = e^t$ or $t = \ln x$

The equation reduces to

$$D(D-1)y - 2y = e^{2t} + e^{-t}$$

$$(D^2 - D - 2)y = e^{2t} + e^{-t}$$

AE: $D^2 - D - 2 = 0$

$(D-2)(D+1) = 0$

$D = 2, \quad D = -1$

$$CF = c_1 e^{2t} + c_2 e^{-t}$$

$$PI = \frac{e^{2t}}{D^2 - D - 2} + \frac{e^{-t}}{D^2 - D - 2}$$

$$= \frac{e^{2t}}{4-2-2} + \frac{e^{-t}}{1+1-2} = \frac{te^{2t}}{2D-1} + \frac{te^{-t}}{2D-1}$$

$$PI = \frac{t\,e^{2t}}{3} + \frac{te^{-t}}{-3}$$

$$CS = y = c_1 e^{2t} + c_2 e^{-t} + \frac{te^{2t}}{3} - \frac{te^{-t}}{3}$$

$$y = c_1 x^2 + \frac{c_2}{x} + \frac{(\ln x)x^2}{3} - \frac{\ln x}{3x}$$

23. Solve $x^2 y'' - 3xy' + 5y = 3\sin(\ln(x))$

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Let $x = e^t \Rightarrow t = \ln x$

Let $D = \frac{d}{dt}$

The given equation is

$$D(D-1)y - 3Dy + 5y = 3\sin(t)$$

$D = \frac{d}{dt}$

$$(D^2 - D - 3D + 5)y = 3\sin(t)$$

$$(D^2 - 4D + 5)y = 3\sin(t)$$

AE: $D^2 - 4D + 5 = 0$

$$D = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm \sqrt{4-5} = 2 \pm i$$

$$CF = (c_1 \cos t + c_2 \sin t)\, e^{2t}$$

$$PI = \frac{3\sin t}{D^2 - 4D + 5} \qquad D^2 = -1$$

$$PI = \frac{3\sin t}{-4D+4} = \frac{3(4+4D)\sin t}{4^2 - 16D^2} = \frac{3(4+4D)\sin t}{16 - 16D^2}$$

$$= \frac{12(1+D)\sin t}{16(1-D^2)} = \frac{\overset{3}{\cancel{12}}(\sin t + \cos t)}{\cancel{4}\,\cancel{16}\,(2)}$$

$$= \frac{3}{8}(\sin t + \cos t) = \frac{3}{8}(\sin(\ln x) + \cos(\ln x))$$

$$PI = \frac{3}{8}(\sin(\ln x) + \cos(\ln x))$$

$$CF = (c_1 \cos(\ln x) + c_2 \sin(\ln x))\, x^2$$

$$y = \frac{3}{8}(\sin(\ln x) + \cos(\ln x)) + (c_1 \cos \ln x + c_2 \sin \ln x)\, x^2$$

24. $x^2y'' - xy' + y = \ln x$, given $y(1) = 0$ and $y'(1) = 0$ 36

Cauchy's form

let $t = \ln x$; $e^t = x$

Let $D = \frac{d}{dt}$

$D(D-1)y - Dy + y = t$

$(D^2 - 2D + 1)y = t$

AE: $(D-1)^2 = 0$

$D = 1, 1$

$CF = (c_1 t + c_2)e^t$

$= (c_1 \ln x + c_2)x$

$PI = \frac{t}{(D-1)^2}$

$= (1-D)^{-2} t$

$= (1+2D)t$

$= (t+2)$

$PI = t + 2 = \ln x + 2$

$y = CF + PI$

$y = c_1 x \ln x + c_2 x + \ln x + 2$

$0 = c_1 \cancel{\ln 1} + c_2 + \cancel{\ln 1} + 2$

$c_2 = -2$

$y' = c_1(\ln x + 1) + c_2 + \frac{1}{x}$

$0 = c_1 - 2 + 1 \Rightarrow c_1 = 1$

$y = x\ln x - 2x + \ln x + 2$

$^nC_r = \frac{n!}{(n-r)!\,r!}$

$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{r(r-1)(r-2)\ldots}$

$(1+x)^n = 1 + {^nC_1}x + {^nC_2}x^2 + \ldots$

$(1+x)^{-2} = 1 + \frac{(-2)}{(1)}x + \frac{(-2)(-3)}{(2)(1)}x^2$

$(1-x)^{-2} = 1 + 2x + 3x^2 + \ldots$

25. $x^2y'' + xy' + y = (\ln x)\sin(\ln x)$

Let $t = \ln x$; $x = e^t$

$D = \frac{d}{dt}$

$D(D-1)y + Dy + y = t\sin t$

$(D^2+1)y = t\sin t$

$D = i, i$

$CF = c_1\cos t + c_2\sin t$

unreliable (use i.m. for failure conditions)

$PI = \frac{t\sin t}{D^2+1}$

replacing D^2 with -1 fails

$PI = t\,\frac{\sin t}{D^2+1} - \frac{2D}{(D^2+1)^2}\sin t$

$= t\,\frac{t\sin t}{2D} - \frac{2D}{(1+D^2)^2}\sin t$

inside this time

$= \frac{t^2}{2}(-\cos t) - \frac{2D}{2(1+D^2)(2D)}t\sin t$

$= \frac{t^2}{2}(-\cos t) +$

$= \frac{t^2}{2}(-\cos t) +$

$= -\frac{t^2}{2}\cos t$

$= -\frac{t^2}{2}\cos t$

$= \frac{-t^2}{2}\cos t$

$$PI = \frac{t\sin t}{D^2+1} \qquad e^{it} = \cos t + i\sin t$$

$$= \text{Im of } \frac{te^{it}}{D^2+1} \qquad \text{shift } D \to D+i$$

$$= \text{Im of } e^{it}\frac{t}{(D+i)^2+1}$$

$$= \text{Im of } e^{it}\frac{t}{D^2+2iD}$$

$$= \text{Im of } e^{it}\frac{t}{(2iD)\left(1+\frac{D}{2i}\right)} = \text{Im of } e^{it}\left(1+\frac{D}{2i}\right)^{-1}\frac{t}{2iD}$$

$$(1-x)^{-1} = 1+x+x^2+\dots$$

$$= \text{Im of } -ie^{it}\left(1-\frac{Di}{2}\right)^{-1}\frac{t}{2D}$$

$$= \text{Im of } -ie^{it}\left(1+\frac{Di}{2}-\frac{D^2}{4}\right)\frac{t}{2D}$$

$$= \text{Im of } -ie^{it}\left(\frac{1}{2D}+\frac{i}{4}-\frac{D}{8}\right)t$$

$$= \text{Im of } \frac{-ie^{it}}{2}\left(\int t\,dt + \frac{it}{2} - \frac{2}{8}\right)$$

$$= \text{Im of } \frac{-ie^{it}}{2}\left(\frac{t^2}{2}+\frac{it}{2}-\frac{1}{4}\right)$$

$$= \text{Im of } \left(\frac{-i\cos t+\sin t}{2}\right)\left(\frac{t^2}{2}+\frac{it}{2}-\frac{1}{4}\right)$$

$$= \text{Im of } \frac{1}{4}(-i\cos t+\sin t)\left(t^2+it-\frac{1}{2}\right)$$

$$= \frac{1}{4}\left(-(\cos t)t^2 + \frac{1}{2}\cos t + t\sin t\right)$$

can be ignored

$$PI = \frac{-t^2\cos t}{4} + \frac{\cos t}{8} + \frac{t\sin t}{4}$$

$$C.S: \; y = CF + PI$$

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26. $(3x+2)y'' + 3(3x+2)y' - 36y = 3x^2+4x+1$

$t = \ln(3x+2)$ $\qquad e^t = 3x+2$

Let $D = \frac{d}{dt}$

$(9)D(D-1)y + 3(3)Dy - 36y = 3\left(\frac{e^t-2}{3}\right)^2 + 4\left(\frac{e^t-2}{3}\right) + 1$

$= \frac{e^{2t}+4-\cancel{4e^t}}{3} + \frac{\cancel{4e^t}-8}{3} + 1$

$= \frac{e^{2t}-4+3}{3} = \frac{e^{2t}-1}{3}$

$(9)D(D-1)y + 3(3)Dy - 36y = \frac{e^{2t}}{3} - \frac{1}{3}$

$(9D^2 - \cancel{9D} + \cancel{9D} - 36)y = \frac{e^{2t}}{3} - \frac{1}{3}$

$9(D^2-4)y = \frac{e^{2t}}{3} - \frac{1}{3}$

AE

$D = 2, -2$

$CF = c_1 e^{2t} + c_2 e^{-2t}$

$PI = \left(\frac{1}{9}\left(\frac{e^{2t}}{3} - \frac{1}{3}\right)\right)\frac{1}{D^2-4}$

$PI = \frac{\left(\frac{e^{2t}}{27} - \frac{1}{27}\right)}{D^2-4}$

$= \frac{1}{27}\frac{e^{2t}}{D^2-4} - \frac{1}{27}\left(\frac{e^0}{D^2-4}\right)$

$= \frac{1}{27}t\left(\frac{e^{2t}}{2D}\right) - \frac{1}{27}\left(\frac{1}{-4}\right)$

$= \frac{t}{27}\left(\frac{1}{2}\right)\left(\frac{e^{2t}}{2}\right) + \frac{1}{27 \times 4}$

$$PI = \frac{t}{54}\left(\frac{e^{2t}}{2}\right) + \frac{1}{27\times 4}$$

$$= \frac{te^{2t}}{108} + \frac{1}{108}$$

$$= \frac{\ln(3x+2)\,(3x+2)^2}{108} + \frac{1}{108}$$

$$y = c_1(3x+2)^2 + c_2(3x+2)^{-2} + \frac{(3x+2)^2}{108}\ln(3x+2) + \frac{1}{108}$$

25.11.19
Monday

27. Solve $(1+x)^2 y'' + (1+x)y' + y = \sin(2(\ln(1+x)))$

$\uparrow ax+b$

Let $1+x = e^t \Rightarrow t = \ln(1+x)$

Let $D = \frac{d}{dt}$

refresh

$$\therefore D(D-1)y + Dy + y = \sin 2t$$

$$(D^2+1)y = \sin 2t$$

A.E: $D^2+1=0$

$$D = i, i$$

$$CF = (c_1\cos t + c_2\sin t) = c_1\cos t + c_2\sin t$$

$$PI = \frac{\sin 2t}{D^2+1} = \frac{\sin 2t}{(-4)+1} = \frac{\sin 2t}{-3}$$

$$C.S = c_1\cos(\ln(1+x)) + c_2\sin(\ln(1+x)) + \frac{\sin 2(\ln(x+1))}{-3}$$

28. Solve $(2x+3)^2 y'' + 6(2x+3)y' + 6y = \ln(2x+3)$

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Let $2x+3 = e^t \Rightarrow t = \ln(2x+3)$

Let $D = \frac{d}{dt}$

$$(4)(D)(D-1)y + (6)(2)(D)y + 6y = t$$

$$(4D^2 - 4D + 12D + 6)y = t$$

$$(4D^2 + 8D + 6)y = t$$

A.E: $4D^2 + 8D + 6 = 0$

$2D^2 + 4D + 3 = 0$

$$D = \frac{-4 \pm \sqrt{16-24}}{4}$$

$$= -1 \pm \frac{\sqrt{-8}}{4}$$

$$= -1 \pm \frac{2\sqrt{2}i}{4} = -1 \pm \frac{\sqrt{2}}{2}i$$

$$D = -1 \pm \frac{i\sqrt{2}}{2}$$

$$CF = \left(c_1 \cos\frac{t}{\sqrt{2}} + c_2 \sin\frac{t}{\sqrt{2}}\right)e^{-t}$$

$$PI = \frac{t}{4D^2+8D+6} = \frac{t}{4(D^2+2D+\frac{3}{2})}$$

$$= \frac{t}{6(\frac{2}{3}D^2 + \frac{4}{3}D + 1)} = \frac{1}{6}\left(1 - \frac{4D}{3}\right)t$$

$$= \frac{t}{6} - \frac{4}{18} = \frac{t}{6} - \frac{2}{9}$$

SERIES SOLUTION OF A DIFFERENTIAL EQUATIONS

- Analytical solutions fail
- Either do numerical integration – Euler (Math III)
- Assume infinite series solution of y (Math II)

Consider a DE of the form

second order linear, homogenous ODE

$$\underbrace{P_0(x)}\frac{d^2y}{dx^2} + \underbrace{P_1(x)}\frac{dy}{dx} + \underbrace{P_2(x)}y = 0 \longrightarrow (1)$$

functions of x

If $P_0(a) \neq 0$, then $x=a$ is called an ordinary point of equation (1)

Otherwise, $x=a$ is a singular point.

A singular point $x=a$ of equation (1) is called regular (analytic) if when equation (1) is put in the form

$$\frac{d^2y}{dx^2} + \frac{Q_1(x)}{x-a}\frac{dy}{dx} + \frac{Q_2(x)}{(x-a)^2}y = 0$$

$Q_1(x)$ and $Q_2(x)$ possess derivates of all orders in the neighbourhood of a $(a-h, a+h)$

A singular point which is not regular is called an irregular singular point.

EXAMPLE (1)

when will this become 0?

Consider the DE $x^2(x+1)^2y'' + (x^2-1)y' + 2y = 0$

$x=0$, $x=-1$ are singular points

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$x=0$ and $x=-1$ are singular points (only highest order term)

Dividing by $x^2(x+1)^2$, we get

$$y'' + \frac{x^2-1}{x^2(x+1)^2}y' + \frac{2y}{x^2(x+1)^2} = 0$$

$$y'' + \boxed{\frac{(x-1)}{x^2(x+1)}}y' + \frac{2}{x^2(x+1)^2}y = 0$$

when $x=0$, $x=-1$ function not defined

Let $P(x) = \frac{x-1}{x^2(x+1)}$, $P'(x) = \frac{2}{x^2(x+1)^2}$

$xP(x) = \frac{x-1}{x(x+1)}$, $x^2P'(x) = \frac{2}{(x+1)^2}$

$x=0$, irregular

This shows that $x=0$ is an irregular singular point

Also, $(x+1)P(x) = \frac{x-1}{x^2}$, $(x+1)^2P'(x) = \frac{2}{x^2}$

defined at $x=-1$

Since these functions are differentiable continuously at $x=-1$, the point $x=-1$ is a regular, singular point.

EXAMPLE (2)

$$x^3(x-1)\frac{d^2y}{dx^2} + 2(x-1)\frac{dy}{dx} + 5xy = 0$$

Study co-ef of highest order term, find singular points

$x^3(x-1) = 0 \Rightarrow x=0, x=1$ are singular points

Dividing by $x^3(x-1)$, we get

$$y'' + \frac{2}{x^3} y' + \frac{5}{x^2(x-1)} y = 0$$

$$P(x) = \frac{2}{x^3} \qquad P'(x) = \frac{5}{x^2(x-1)}$$

test at $x=0$

$$x P(x) = \frac{2}{x^2} \qquad x^2 P(x) = \frac{5}{(x-1)}$$

$x=0$ is irregular, singular point

test at $x=1$

$$(x-1) P(x) = \frac{2(x-1)}{x^3} \qquad (x-1)^2 P'(x) = \frac{5(x-1)}{x^2}$$

differentiable at $x=1$

$x=1$ is a regular, singular point

EXAMPLE (3)

$$2x^2y'' + 7x(x+1)y' - 3y = 0$$

singular points: $x=0$

Dividing by $2x^2$,

$$y'' + \frac{7(x+1)}{2x} y' - \frac{3}{2x^2} y = 0$$

$$P(x) = \frac{7(x+1)}{2x} \qquad P'(x) = \frac{3}{2x^2}$$

$$x P x = \frac{7(x+1)}{2} \qquad x^2 P'(x) = \frac{3}{2}$$

$\therefore$ 0 is a regular, singular point

Theorem 1

When $x=a$ is an ordinary point of eq. (1), its every solution can be expressed in the form

$$y = a_0 + a_1(x-a) + a_2(x-a)^2 + \dots \quad \text{—— (2)}$$

Theorem 2

When $x=a$ is a regular point of eq (1), at least one of the solutions can be expressed as

$$y = (x-a)^m \left[a_0 + a_1(x-a) + a_2(x-a)^2 + \dots\right] \quad \text{—— (3)}$$

SERIES SOLUTION WHEN $x=0$ IS AN ORDINARY POINT OF THE EQUATION

Consider $P_0 \frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0$

where P's are polynomials in x and $P_0(0) \neq 0$

To solve this equation,

(i) Assume its solution to be of the form

$$y = a_0 + a_1 x + a_2 x^2 + \cdots$$

(ii) Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and substitute in the given eq.

(iii) Equate to 0 the coefficients of the various powers of x, and determine $a_2, a_3 \ldots$ in terms of a_0 and a_1 — 2 arbitrary constants

(iv) Substitute the values of a_2, a_3 and so on to get the desired series solution having a_0 and a_1 as its arbitrary constants.

29. Solve in series the equation $\frac{d^2y}{dx^2} + xy = 0$

coefficient of $\frac{d^2y}{dx^2} \neq 0$ when $x = 0$

The coefficient of $y'' = 1 \neq 0$ at $x = 0$

Therefore, $x = 0$ is an ordinary point of the DE.

Theorem (1) applies when $x = 0$ is ordinary.

Assume $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \infty$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots + n a_n x^{n-1} + \cdots \infty$$

$$y'' = 2a_2 + 3\cdot 2\cdot a_3 x + 4\cdot 3\cdot a_4 x^2 + \cdots + n(n-1) a_n x^{n-2} + \cdots \infty$$

Substituting for y'' and y

The given DE is

$$(2\cdot1\cdot a_2 + 3\cdot2a_3x + 4\cdot3\cdot a_4x^2 + \dots + n(n-1)a_nx^{n-2} + \dots\infty) + x(a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n + \dots\infty) = 0$$

$$2\cdot1\cdot a_2 + x(a_0 + 3\cdot2\cdot a_3) + x^2(a_1 + 4\cdot3a_4) + \dots + x^n(a_{n-1} + (n+2)(n+1)a_{n+2}) = 0$$

Equating to 0 to coefficients of various powers of x, we get

$$2\cdot1\cdot a_2 = 0 \Rightarrow \boxed{a_2 = 0}$$

$$a_0 + 3\cdot2\,a_3 = 0 \Rightarrow \boxed{a_3 = \frac{-a_0}{3!}}$$

$$4\cdot3\cdot a_4 + a_1 = 0 \Rightarrow a_4 = \frac{-a_1}{4\cdot3} \Rightarrow \boxed{a_4 = \frac{-2a_1}{4!}}$$

$$5\cdot4a_5 + a_2 = 0 \Rightarrow \boxed{a_5 = 0} \quad (\because a_2 = 0)$$

$\therefore a_8, a_{11}, a_{14} \dots$ are all 0.

In general,

$$(n+2)(n+1)a_{n+2} + a_{n-1} = 0$$

recurrence relation ✓

$$a_{n+2} = \frac{-a_{n-1}}{(n+2)(n+1)}, \quad n = 1, 2, 3 \dots$$

Substituting $n=4,5,6$ and so on successively

$$a_6 = \frac{-a_3}{6\cdot 5} = \frac{a_0}{6\cdot 5\cdot 3!} = \boxed{\frac{4a_0}{6!} = a_6}$$

$$a_7 = \frac{-a_4}{7\cdot 6} = \frac{2a_1}{7\cdot 6\cdot 4!} = \boxed{a_7 = \frac{2\cdot 5a_1}{7!}}$$

$$\boxed{a_8 = 0}$$

$$a_9 = \frac{-a_6}{9\cdot 8} = \frac{-4a_0}{9\cdot 8\cdot 6!} = \boxed{a_9 = \frac{-4\cdot 7a_0}{9!}}$$

$$\therefore y = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots \infty$$

$$= a_0 + a_1x + \left(\frac{-a_0}{3!}\right)x^3 + \left(\frac{-2a_1}{4!}\right)x^4 + \left(\frac{4a_0}{6!}\right)x^6 + \left(\frac{2\cdot 5a_1}{7!}\right)x^7 + \left(\frac{-4\cdot 7a_0}{9!}\right)x^9 + \dots$$

$$= a_0\left(1 - \frac{x^3}{3!} + \frac{1\cdot 4x^6}{6!} - \frac{1\cdot 4\cdot 7x^9}{9!} + \dots \infty\right)$$

$$+ a_1\left(x - \frac{2x^4}{4!} + \frac{2\cdot 5}{7!}x^7 + \dots \infty\right)$$

Frobenius Method: Series solution when $x=0$ is a regular singular point

coefficient of highest-order derivative $=0$

consider $P_0 \frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = 0$

becomes 0 at $x=0$ (if $\neq 0$, use prev. method)

Using theorem(2)

(i) Assume the solution to be

$$y = x^m (a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots \infty)$$

(ii) Find y', y'' and substitute

(iii) Equate to 0 the coefficient of the lowest degree term in x.

It gives a quadratic equation called the indicial equation.

(iv) Equate to zero the coefficients of the other powers of x and find the values of $a_1, a_2, a_3 \dots$ in terms of a_0

(v) The complete solution depends on the nature of roots of the indicial equation

(vi) When the roots of the indicial equation are distinct and do not differ by an integer, the complete solution is

y evaluated at m_1

$$y = c_1 (y)_{m_1} + c_2 (y)_{m_2}$$

a constants as $c_1 a_0$ & $c_2 a_0$ are constants

where m_1 and m_2 are the roots

30. Solve in series equation

$$9x(1-x)y'' - 12y' + 4y = 0$$

$x=0$ is a singular point since coefficient of y'' is 0 at $x=0$.

Therefore, assume

$$y = x^m(a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots \infty)$$

$$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \dots + a_nx^{m+n} + \dots \infty$$

$$y' = ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \dots + (m+n)a_nx^{m+n-1} + \dots \infty$$

$$y'' = m(m-1)a_0x^{m-2} + (m+1)(m)a_1x^{m-1} + (m+2)(m+1)a_2x^m + \dots \infty$$

Substituting,

$$9x(1-x)\left[m(m-1)a_0x^{m-2} + (m+1)(m)a_1x^{m-1} + (m+2)(m+1)a_2x^m + \dots \infty\right]$$

$$-12\left[ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \dots \infty\right]$$

$$+4\left[x^ma_0 + a_1x^{m+1} + a_2x^{m+2} + \dots \infty\right]$$

The lowest power of x is x^{m-1}

Equating to 0 its coefficient, we get

$$9x\,m(m-1)a_0x^{m-2} - 12ma_0x^{m-1}$$

$$[9m(m-1)a_0 - 12ma_0) = 0$$

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$$9m^2a_0 - 9ma_0 - 12ma_0 = 0$$

$$9m^2 - 21m = 0 \quad (a_0 \neq 0)$$

$$3m(3m-7) = 0$$

$m = 0$ and $m = 7/3$ (difference should not be an integer and are distinct)

Equating to 0 the coefficient of x^m

$$-9(m)(m-1)a_0 + 9(m+1)(m)a_1 + (-12)(m+1)a_1 + 4a_0 = 0$$

$$(-9m^2 + 9m)a_0 + 9(m^2+m)a_1 - (12m+12)a_1 + 4a_0 = 0$$

$$a_0(-9m^2 + 9m + 4) + a_1(9m^2 + 9m - 12m - 12) = 0$$

$$-a_0(9m^2 - 9m - 4) + a_1(9m^2 - 3m - 12) = 0$$

$$-a_0(9m^2 - 12m + 3m - 4) + a_1(3)(3m^2 - m - 4) = 0$$

$$-a_0(3m(3m-4) + 1(3m-4)) + 3a_1(3m^2 - 4m + 3m + 4) = 0$$

$$-a_0(3m+1)(3m-4) + 3a_1(3m-4)(m+1) = 0$$

$$m \neq 4/3 \quad (m = 0, 7/3)$$

$$-a_0(3m+1) + 3a_1(m+1) = 0$$

$$3a_1(m+1) = a_0(3m+1)$$

Similarly,

$$3a_2(m+2) = a_1(3(m+1)+1)$$

$$3a_3(m+3) = a_2(3(m+2)+1)$$

$$a_1 = \frac{3m+1}{3(m+1)}a_0, \quad a_2 = \frac{3m+4}{3(m+2)}a_1 = \frac{(3m+4)(3m+1)}{3(m+2)(3)(m+1)}$$

$$a_3 = \frac{3m+7}{3(m+3)} a_2 = \frac{(3m+7)(3m+4)(3m+1)}{3^3 (m+3)(m+2)(m+1)} a_0$$

and so on

when $m=0$, $a_1 = \frac{a_0}{3}$

$$a_2 = \frac{1\cdot 4}{3^2 \cdot 1\cdot 2} a_0 = \frac{1\cdot 4}{3\cdot 6}$$

$$a_3 = \frac{1\cdot 4\cdot 7}{3^3\cdot 1\cdot 2\cdot 3} a_0 = \frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9} a_0$$

when $m = 7/3$,

$$a_1 = \frac{8}{10} a_0, \quad a_2 = \frac{8\cdot 11}{10\cdot 13} a_0, \quad a_3 = \frac{8\cdot 11\cdot 14}{10\cdot 13\cdot 16} a_0$$

The complete solution is

$$y = c_1 (y)_{m_1} + c_2 (y)_{m_2}$$

$$= c_1 \left[1 + \frac{1}{3} x + \frac{1\cdot 4}{3\cdot 6} x^2 + \frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9} x^3 \cdots \right] a_0 x^0$$

$$+ c_2 \left[1 + \frac{8}{10} x + \frac{8\cdot 11}{10\cdot 13} x^2 + \frac{8\cdot 11\cdot 14}{10\cdot 13\cdot 16} x^3 \cdots \right] a_0 x^{7/3}$$

But $c_1 a_0$ is one constant and $c_2 a_0$ is another.